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# High Resolution Capture Zone Delineation for Hydrological Systems

CGS Seed Project Final Report

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Groundwater Studies

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## Summary

The practical management of urban groundwater resources is complicated by the presence of pollutants. Pollutant sources may be distributed broadly over an aquifer through land-use practices or contaminated precipitation, or be more spatially confined, occurring as point sources such as industrial sites and spillages. Tracking contaminants in aquifers is a required part of the safe administration of public water production bores. This project attempts to develop accurate *predictive* methods for determining geometries and flow pathways for contaminant distributions in aquifers. Attenuation processes, such as chemical reaction, dispersion or adsorption, are not considered in this report.

Typically, predictions of flow paths in aquifer systems are made via numerical solution of the appropriate hydrological equations. For certain systems, notably the lake-aquifer systems which characterise the Swan Coastal Plain, it is difficult to construct numerical models with sufficiently high resolution to map accurate lake capture and release zones. Furthermore, it is equally difficult to gauge the true accuracy of any existing numerical model of the system.

An alternative method, examined in this project, is to determine *exact* flow paths for relevant hydrological systems at steady state, independent of numerical simulations. The method is based on potential theory for steady systems, and can yield exact algebraic formulae for flow paths. In principle, these exact flow paths provide resolutions far beyond those achievable by numerical simulations, allowing modellers to test the validity of their programs for problematic systems involving pumping wells, rivers and lakes. This knowledge will flow into field applications, allowing groundwater resource managers to make decisions based on higher quality flow path information.

Research outcomes include the identification and elaboration of diverse mathematical tools relevant to the integration of the flow path equations, the solution of several simple test cases, and successful application to arbitrary arrangements of pumping wells in two dimensions. The flow path equations for lake-aquifer systems have been derived explicitly, but their ultimate solution requires further substantial effort.

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## 1. Introduction

The management of water resources and associated production systems is fraught with difficulties associated with the encroachment of pollutant plumes on production bores, wetlands and other sensitive areas. By moving with the regional groundwater flow these plumes can cover large distances, and have the potential to cause serious disruption and damage to water supplies, and to impact users and ecological systems. In particular, the interaction of pollutant plumes with production bores and water bodies of recreational and/or environmental significance is of high concern for water resource managers. Detailed knowledge of the hydrological systems surrounding such key sites is critical to the problem of water resource protection.

This project is concerned with developing high resolution methods for calculating water flow trajectories near pumping bores and surface water bodies. In turn, these flow paths define the capture zones for the associated bores and water bodies, which can be used to develop production bore protection and wetland management plans. Recent studies (Townley *et al.*, 1993; Trefry and Townley, 1996; Trefry and Townley, 1998; Townley and Trefry, 1998) have highlighted both the three-dimensional nature of flow paths, and the extreme difficulty of determining accurate numerical capture zones, for lake-aquifer systems. For example, using common finite element or finite difference techniques inevitably leads to discretisation errors in describing the curvature of boundaries of the target surface water bodies. Similar problems are encountered in the description of well geometries. Yet it is the detailed properties of these boundaries, and the gradients of the associated fluid flow fields, that are critical to determining capture/release zone geometries in the far-field. These problems are magnified when the simulated domains have very different length scales, or significant anisotropy. The former case pertains to the Swan Coastal Plain, where the length scales of wetlands and their capture zones are typically one or two orders of magnitude greater than the average depth of the underlying surficial aquifer.

It may be possible, through the use of exact mathematical techniques, to avoid these numerical problems in the calculation of capture/release zone geometries. Central to this approach is the availability of suitable exact flow potentials for the target systems. Whilst this availability is by no means assured for all possible hydrological systems, several key exact flow potentials have been identified. These include potentials for pumping wells in aquifers (Bear and Jacobs, 1965; Javandel and Tsang; 1986), a pumping well near a river (Wilson 1991), and for flow near a circular lake overlying a

deep aquifer (Trefry and Townley 1997). The latter exact potential can be extended to provide a formally exact flow potential for a lake overlying a shallow aquifer, which is relevant to the Swan Coastal Plain.

This project has tested the application of several theoretical techniques for deriving exact flow paths (without involving numerical models) from these exact flow potentials. The techniques involve the algebraic solution of coupled ordinary differential parametric equations for the flow path fields. For stagnation-free flow, the coupled system reduces in dimensionality to an explicit system, which may admit solutions via various independent mathematical methods. These solutions are flow paths for the system; capture zones are limiting subsets of the flow path solutions.

In succeeding sections, the mathematical tools used in the project are described and demonstrated by example. Then flow paths for several two-dimensional (plan projection) systems containing pumping wells are derived explicitly. Finally, the groundwater flow path equations for a shallow lake system are presented, before concluding remarks are made.

## 2. Flow Potentials and Flow Paths

In this section the basic approach to the calculation of high resolution capture zones is mapped out. Consider an aquifer system represented in the three-dimensional coordinates  $(x,y,z)$ . If the boundary conditions and material properties of the system are independent of time, the system will tend to a steady state. For a groundwater flow problem, the steady state will define a distribution of heads throughout the three-dimensional space. This head distribution is referred to as the *flow potential* for the system, and is normally denoted by  $\phi$ . From Darcy's Law, the flow velocity vector  $v$  is related to  $\phi$  by (Bear and Jacobs, 1965):

$$v = v(x, y, z) = - \frac{K}{\theta} \cdot \nabla \phi(x, y, z) \quad (2.1)$$

In equation (2.1)  $K$  is the hydraulic conductivity tensor,  $\theta$  is the aquifer porosity and  $\nabla$  is the gradient operator. If the coordinate axes are chosen to coincide with the principal components of  $K$ , equation (2.1) can be considered to be a set of three simultaneous equations arising from the three spatial components:

$$\begin{aligned}
v_x &= -\frac{K_{xx}}{\theta} \frac{\partial \phi}{\partial x} \\
v_y &= -\frac{K_{yy}}{\theta} \frac{\partial \phi}{\partial y} \\
v_z &= -\frac{K_{zz}}{\theta} \frac{\partial \phi}{\partial z}
\end{aligned}
\tag{2.2}$$

Noting that velocity is simply the time rate of change of position, equation (2.2) immediately becomes

$$\begin{aligned}
\frac{dx}{dt} &= -\frac{K_{xx}}{\theta} \frac{\partial \phi}{\partial x} \\
\frac{dy}{dt} &= -\frac{K_{yy}}{\theta} \frac{\partial \phi}{\partial y} \\
\frac{dz}{dt} &= -\frac{K_{zz}}{\theta} \frac{\partial \phi}{\partial z}
\end{aligned}
\tag{2.3}$$

Equations (2.3) show that Darcy's Law leads naturally to a coupled set of simultaneous first order differential equations, since  $\phi$  is a function of  $x$ ,  $y$  and  $z$ . Note that the right hand sides of equations (2.3) are independent of time (autonomic), since the system is assumed to be at steady state. The solutions of these equations are *flow paths* for the system, representing the position (as a function of time  $t$ ) of a notional point particle flowing through the head distribution. The flow paths determined from equation (2.3) are specified to within three constants of integration. These may conveniently be fixed in terms of the initial position of the point particle whose flow path is to be determined.

Solutions to equation (2.3) are parameterised by the travel time variable  $t$ , which allows velocities to be calculated at every point on the flow path. If *shapes* of flow paths are more important than rates of movement along them, then equation (2.3) can be simplified to remove the parameterisation by  $t$ . For example, if  $v_x$  is everywhere non-zero (stagnation free),

$$\begin{aligned}
\frac{dy}{dx} &= \frac{K_{yy}}{K_{xx}} \frac{\partial \phi / \partial y}{\partial \phi / \partial x} \\
\frac{dz}{dx} &= \frac{K_{zz}}{K_{xx}} \frac{\partial \phi / \partial z}{\partial \phi / \partial x}
\end{aligned}
\tag{2.4}$$

The solutions to equations (2.4) are the  $y$  and  $z$  components of the flow paths, written as functions of the  $x$  component. The travel time variable has been removed. Clearly,

equations (2.4) can also be recast with either  $y$  or  $z$  as the independent variable, instead of  $x$ . The choice of variable to use is determined by the characteristics of the potential  $\phi$ . For example, if  $\phi$  contains a regional gradient in the direction of  $x$  then it would be natural to use  $x$  as the dependent variable.

Equations (2.3) and (2.4) provide a formal framework for determining exact flow paths for any steady flow potential  $\phi$ . If velocity information is important, *e.g.* for the calculation of travel times and flux rates along flow paths, the formulation of equations (2.3) should be solved to provide position solutions explicitly dependent on time. If velocity information is not important, the formulation of equation (2.4) should be considered as it may provide a simpler set of equations to be solved. Note that the above method is limited to steady flow; other formulations exist for transient flow problems (see Matanga (1993) and references therein).

Application of the above methods for calculating exact flow paths depends upon the availability of suitable algebraic potentials. The following section discusses some candidate potential solutions available in the literature.

### **3. A Short Review of Analytic Flow Potentials**

A large variety of algebraic flow potentials are available. In this early proof-of-concept stage of the project, attention is confined to those potentials for which the spatial derivatives are simple enough to provide relatively straightforward solutions to the flow path equations. Two-dimensional potentials for pumping well systems have relatively simple algebraic potentials (Bear and Jacobs, 1965; Strack, 1989) and their capture zones have been studied extensively by algebraic means (Bear and Jacobs, 1965; Javandel and Tsang, 1986; Grubb, 1993). Wilson (1993) extended this work to systems bounded by linear water bodies. In all these cases, capture zone geometries were inferred from transcendental parameterisations of the position coordinates in terms of  $t$ .

Relevant steady state algebraic potentials for three-dimensional systems and lake-aquifer systems are more scarce. Tóth (1962) applies a result from heat conduction theory (Carslaw and Jaeger, 1959; see also Riesenkampf and Kalinin, 1941) to a system of ellipsoidal conducting inclusions embedded in a three-dimensional medium. Trefry and Townley (1997) show that the Carslaw and Jaeger result can be transformed into an exact potential for an idealised three-dimensional lake-aquifer system. Two-dimensional

analogues of the Trefry and Townley potential were reported by Copson (1947) in an electrostatic context.

For the Swan Coastal Plain, hydrological systems involving shallow lakes, pumping wells and ambient regional flow are most relevant. Thus the works of Copson (1947), Grubb (1993), Wilson (1993), and Trefry and Townley (1997) are of principal interest.

## 4. Tools for Integrating Flow Path Equations

The flow path equations (equations (2.3) and (2.4)) are written in terms of formal derivatives of the flow potential  $\phi$ . In general,  $\phi$  is a non-linear function of the position coordinates. In turn, the flow path equations will take the form of systems of non-linear coupled differential equations, for which no guaranteed method of solution exists. Where the algebraic form of  $\phi$  is such that the flow path equations have no obvious direct solutions, other solution techniques must be attempted. This section describes several techniques for integrating the derived flow path equations. First, the standard Frobenius series expansion method is discussed, together with a useful adjunct special function technique for the Frobenius method. Then the application of Lie methods to the flow path equations is outlined, followed by a brief discussion of differential polynomial techniques. Finally, the application of complex potential theory is included as a parallel approach to integrating the flow path equations.

### 4.1 The Frobenius Method

The Frobenius method is a standard solution technique for differential equations involving variable coefficients (Boyce and DiPrima, 1977). The method is explained by way of a simple one-dimensional example; generalisations to higher dimensionality are straightforward. Consider the ordinary differential equation:

$$\frac{dy}{dx} + \sin(x)y = 0 \tag{4.1}$$

The solution to this equation is  $y = A \exp(\cos x)$ , where  $A$  is an integration constant. The Frobenius method involves expanding the solution heuristically as a power series in  $x$ :

$$y(x) = a_0 + a_1x + a_2x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n \quad (4.2)$$

Substituting equation (4.2) into equation (4.1) yields

$$\sum_{n=0}^{\infty} (n+1)a_{n+1}x^n + \left( \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} \right) \left( \sum_{n=0}^{\infty} a_n x^n \right) = 0 \quad (4.3)$$

where the first bracketed term is the Maclaurin series expansion for  $\sin x$ . Equation (4.3) can be expanded and like powers of  $x$  grouped together. Since  $x$  is unspecified, and the right hand side of equation (4.3) is zero, the coefficient of each power of  $x$  must itself be equal to zero. This provides the following set of recurrence relations for the expansion coefficients  $a_n$ :

$$\begin{aligned} x^0 : a_1 &= 0 \\ x^1 : 2a_2 + a_0 &= 0 \\ x^2 : a_1 + 3a_3 &= 0 \\ x^3 : -\frac{1}{6}a_1 + a_2 + 4a_4 &= 0 \end{aligned} \quad (4.4)$$

and so on. It is simple to show from equation (4.4) that every odd coefficient is zero, and that the first few terms of the solution for  $y$  are:

$$y(x) = a_0 \left( 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{31x^6}{720} + \frac{379x^8}{40320} - \dots \right) \quad (4.5)$$

Whilst this is a formal solution to equation (4.1), it would be more useful if the closed form solution  $y = A \exp(\cos x)$  could be recaptured from equation (4.5). If the numerical forms of the coefficients in equation (4.5) were simpler, analytical forms for the coefficients  $a_{2n}$  as explicit functions of the summation index  $n$  could be deduced by induction. For example, the denominators in equation (4.5) appear to be related in some way to the factorial function. The origins of the numerators are more obscure. In general this series term identification presents a serious obstacle to the derivation of closed form Frobenius solutions to differential equations. In the following subsection a method for identifying hypergeometric representations of power series terms is described, giving the possibility of generating exact closed form expressions for Frobenius solutions for a wide range of linear and non-linear differential equations.

## 4.2 Minimal Hypergeometric Representations

A wide variety of elementary and special functions can be represented in terms of the generalised hypergeometric function,  ${}_pF_q$ , defined by (Erdelyi *et al.*, 1953):

$${}_pF_q(\alpha_1, \dots, \alpha_p; \gamma_1, \dots, \gamma_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \dots (\alpha_p)_k}{(\gamma_1)_k \dots (\gamma_q)_k} \frac{x^k}{k!} \quad (4.6)$$

where the Pochhammer symbols  $(a)_n = \Gamma(a+n)/\Gamma(a) = (a)(a+1)\dots(a+n-1)$ , and  $\Gamma$  is the Gamma function (Abramowitz and Stegun, 1965). From equation (4.6) it can be seen that  ${}_pF_q$  is a power series. The problem of identifying a generalised hypergeometric representation for a specific power series reduces to deducing the appropriate numbers ( $p$  and  $q$ ) and values of parameters  $\alpha, \gamma$  to be inserted into equation (4.6). From equation (4.6) it can be seen that any hypergeometric  ${}_pF_q$  can be expressed in terms of a higher hypergeometric  ${}_mF_n$  where  $m > p$  and  $n > q$  simply by introducing  $\alpha$  and  $\gamma$  Pochhammer parameters that cancel each other. In this report attention is focussed on minimal hypergeometrics, *i.e.* hypergeometric functions where redundant Pochhammer symbols have been eliminated.

The following procedure (Roach, 1992) is useful for deducing equivalent minimal hypergeometric forms from power series coefficients. Consider the power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad (4.7)$$

If the ratios of successive coefficients  $r_n = (n+1)a_{n+1}/a_n$  are rational functions in  $n$ , and can be written

$$r_n = (n+1) \frac{a_{n+1}}{a_n} = c \frac{\prod_{i=1}^p (n + \mu_i)}{\prod_{i=1}^q (n + \nu_i)} \quad (4.8)$$

for  $\mu_i, \nu_i$  and  $c$  independent of  $n$ , then an immediate identification can be made:

$$y(x) = y_0 {}_pF_q(\mu_1, \dots, \mu_p; \nu_1, \dots, \nu_q; cx) \quad (4.9)$$

where  $y_0$  is a multiplying constant. The problem is to determine the  $\mu_i$ ,  $\nu_i$  and  $c$  parameters in equation (4.9). If  $r_n$  can be expressed as

$$r_n = \frac{P(n)}{Q(n)} \quad (4.10)$$

where  $P(n)$  and  $Q(n)$  are non-zero finite polynomials in  $n$ ,

$$P(n) = \sum_{i=0}^p p_i n^i \quad ; \quad Q(n) = 1 + \sum_{i=0}^q q_i n^i \quad (4.11)$$

then equation (4.11) leads to a system of simultaneous linear equations for  $p_i$  and  $q_i$ :

$$\sum_{i=0}^p p_i n^i - r_n \sum_{i=1}^q q_i n^i = r_n \quad (4.12)$$

since  $r_n Q(n) - P(n) = 0$  for  $n = 1, 2, \dots$ . As an example, consider the case where  $P$  and  $Q$  are quadratic in  $n$ . Then, using  $p = q = 2$  and equation (4.12) leads to the matrix system

$$\begin{pmatrix} 1 & 1 & 1 & -r_1 & -r_1 \\ 1 & 2 & 4 & -2r_2 & -4r_2 \\ 1 & 3 & 9 & -3r_3 & -9r_3 \\ 1 & 4 & 16 & -4r_4 & -16r_4 \\ 1 & 5 & 25 & -5r_5 & -25r_5 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} \quad (4.13)$$

Equation (4.13) is exactly soluble, yielding exact rational functions ( $P$  and  $Q$ ) of degree 2. If  $P$  and  $Q$  can then be factorised into simple product forms, as in equation (4.8), the appropriate hypergeometric function can be written down immediately. Then, standard relationships between hypergeometric and elementary functions (Erdelyi *et al.*, 1954; Roach, 1996) can be used to deduce closed form expressions for the initial power series.

### 4.3 Symmetry Group Methods

Many differential equations, and systems of differential equations, are invariant under particular coordinate transformations, *i.e.* the equations retain their original form after

the coordinate transformation has been carried out. For example, the first order differential equation

$$y^2 \frac{dy}{dx} = \frac{y}{x^3} + \frac{4y^2}{x^2} \quad (4.14)$$

is invariant under the transformations  $x_1 = e^\varepsilon x$  and  $y_1 = e^{-\varepsilon} y$ , regardless of the value of the single parameter  $\varepsilon$ . This can be readily be seen by substituting for  $x$  and  $y$  in equation (4.14), yielding

$$e^{2\varepsilon} y_1^2 \frac{e^\varepsilon}{e^{-\varepsilon}} \frac{dy_1}{dx_1} = \frac{e^\varepsilon}{e^{-3\varepsilon}} \frac{y_1}{x_1^3} + \frac{e^{2\varepsilon}}{e^{-2\varepsilon}} \frac{4y_1^2}{x_1^2} \quad (4.15)$$

The exponential factors cancel completely, leaving an equation identical to equation (4.14), but expressed in terms of  $x_1$  and  $y_1$ , rather than  $x$  and  $y$ . This invariance provides a means of integrating the normally intractable equation (4.14). The form of the one-parameter transformation prompts the change of dependent variable to  $u(x,y) = x y$ . Equation (4.14) then becomes

$$x u^2 \frac{du}{dx} = u^3 + 4u^2 + u \quad (4.16)$$

which integrates directly to give

$$\sqrt{u(u+4)+1} e^{\frac{2}{\sqrt{3}} \tanh^{-1}\left(\frac{u+2}{\sqrt{3}}\right)} = Cx \quad (4.17)$$

where  $C$  is an integration constant. The implicit solution for the differential equation (4.14) is then immediately recovered by substituting  $u = x y$  in equation (4.17). Clearly, knowledge of the invariants of a differential equation can be a great help in moving toward a solution. However for more complicated equations the invariant forms are not obvious. This is the basis of Lie's fundamental problem (Lie, 1874; Hill, 1982), how to find a one-parameter group of coordinate transformations that leaves a given first order differential equation invariant. This problem remains unsolved. In the present application, the time-implicit flow path equations (2.4) fall into the unsolved category since they take the form of first-order coupled equations explicit in the dependent and independent variables (*i.e.* non-autonomic). Fortunately, for autonomic differential systems, such as the time-explicit flow equations (2.3), the commutation theorem of Lie

series (Hill, 1982) provides an automatic formal solution. Consider the two-dimensional system

$$\begin{aligned}\frac{dx}{dt} &= F(x, y) \\ \frac{dy}{dt} &= G(x, y)\end{aligned}\tag{4.18}$$

where  $x(0) = \alpha$  and  $y(0) = \beta$  are the flow path starting points. The commutation theorem then provides the formal solutions

$$\begin{aligned}x(t) &= e^{tM} \alpha \\ y(t) &= e^{tM} \beta\end{aligned}\tag{4.19}$$

where the Lie operator  $M$  is defined by

$$M = F(\alpha, \beta) \frac{\partial}{\partial \alpha} + G(\alpha, \beta) \frac{\partial}{\partial \beta}\tag{4.20}$$

and the exponential term is expanded by the Cayley-Hamilton operator theorem

$$e^{tM} = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n\tag{4.21}$$

The notation  $M^n$  indicates operator composition, *i.e.*  $M^2 = M(M(\cdot))$ . As an illustrative example, consider the following single first-order non-linear differential equation

$$\frac{dx}{dt} = -x^{3/2}\tag{4.22}$$

The operator  $M$  is then given by  $M = -\alpha^{3/2} \partial/\partial \alpha$ , where  $\alpha$  is the initial value of  $x(t)$ . Compositions of  $M$  then satisfy  $M(\alpha) = -\alpha^{3/2}$ ,  $M^2(\alpha) = 3\alpha^2/2$ ,  $M^3(\alpha) = -3\alpha^{5/2}$  and so on. In general,  $M^n(\alpha) = (-1)^n (n+1)! \alpha^{(n+2)/2}/2^n$ , hence by equation (4.19) the solution is

$$\begin{aligned}x(t) &= e^{tM} \alpha = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n(\alpha) \\ &= \sum_{n=0}^{\infty} \frac{(-t)^n (n+1) \alpha^{(n+2)/2}}{2^n} = \frac{4\alpha}{(\sqrt{\alpha} t + 2)^2}\end{aligned}\tag{4.23}$$

Again, as for the Frobenius method, determination of an explicit form of the solution comes down to the exact summation of an algebraic series. There are several computer implementations of Lie commutation and other symmetry methods for solving non-linear differential equations (*e.g.* Head, 1997).

#### 4.4 Complex Potential Theory

In two-dimensional problems, complex function theory provides an elegant method for deducing exact flow paths for steady potentials. Consider the complex function  $f(z)$ , where  $z = x + iy$ , which can be decomposed into real ( $\phi$ ) and imaginary ( $\psi$ ) parts:

$$f(z) = f(x + iy) = \phi(x, y) + i\psi(x, y) \quad (4.24)$$

If  $\phi(x, y)$  can be identified with the steady flow potential for a two-dimensional flow problem, then  $\psi(x, y)$  is the corresponding streamfunction;  $f(z)$  is said to be the complex potential for the problem. Hence, given an explicit form for  $f(z)$ , the exact streamfunction can be constructed simply by taking the imaginary part of  $f(z)$ . Flow paths are then determined by contours of the streamfunction. As an example, consider the flow potential (Bear and Jacobs, 1965)

$$f(z) = \frac{Q}{2\pi} \ln z \quad (4.25)$$

Noting that  $z = x + iy = r e^{i\theta}$ , where  $r = (x^2 + y^2)^{1/2}$  is the modulus (or amplitude) of  $z$ , and  $\theta = \tan^{-1}(y/x)$  is the argument (or phase) of  $z$ , we have

$$f(z) = \frac{Q}{2\pi} \left[ \frac{1}{2} \ln(x^2 + y^2) + i \tan^{-1} \frac{y}{x} \right] \quad (4.26)$$

Thus, from equation (4.24), the identifications

$$\phi(x, y) = \frac{Q}{4\pi} \ln(x^2 + y^2) \quad \text{and} \quad \psi(x, y) = \frac{Q}{2\pi} \tan^{-1}(y/x) \quad (4.27)$$

can be made. In equation (4.27),  $\phi(x, y)$  is congruent with the potential for a point source (injection well) in a two-dimensional (in plan) aquifer, hence  $\psi(x, y)$  represents the streamfunction for this system. Equation (4.25) then gives the complex potential for

a single injection well ( $Q > 0$ ) or pumping well ( $Q < 0$ ). Contours of  $\psi$  give flow paths for the single well system.

In most circumstances where explicit algebraic forms for the flow potential  $\phi$  are available, corresponding algebraic forms for the complex potential  $f$  are lacking, *i.e.* it is usually not obvious how to construct  $f$  given only the real part of  $f$ . In some circumstances this difficulty can be overcome using the Cauchy-Riemann equations (Marsden, 1973) for analytic complex functions. These state that if  $f(z)$  of equation (4.24) is differentiable, then

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= \frac{\partial\psi}{\partial y} \\ \frac{\partial\phi}{\partial y} &= -\frac{\partial\psi}{\partial x}\end{aligned}\tag{4.28}$$

Thus, given an algebraic form for  $\phi(x,y)$ ,  $\psi(x,y)$  is determined by the system of partial differential equations (4.28). As an example, consider the saddle potential

$$\phi(x,y) = x^2 - y^2\tag{4.29}$$

The Cauchy-Riemann equations then reduce to

$$\begin{aligned}\frac{\partial\psi}{\partial y} &= 2x \\ \frac{\partial\psi}{\partial x} &= -2y\end{aligned}\tag{4.30}$$

Integration of these equations is trivial, resulting in the streamfunction  $\psi(x,y) = 2xy$ , whence it can be shown that the complex potential appropriate for the flow potential of equation (4.29) is

$$f(z) = f(x + iy) = z^2\tag{4.31}$$

The Cauchy-Riemann approach does not always succeed in determining streamfunctions, especially for non-separable flow potentials  $\phi$ . As a simple example of the failure of the Cauchy-Riemann approach, consider the parabolic potential

$$\phi(x,y) = x^2 + y^2\tag{4.32}$$

It can easily be seen that the Cauchy-Riemann equations for  $\psi$  lead to unreconcilable solutions.

#### 4.5 Differential Algebra - Differential Polynomials and Rings

For some classes of flow potentials, the time-implicit flow path equations take the form

$$\frac{dy}{dx} = R(y, x) \quad (4.33)$$

where  $R$  is a polynomial function. This is a subset of a more general class of problem,

$$P(y', y, x) = 0 \quad (4.34)$$

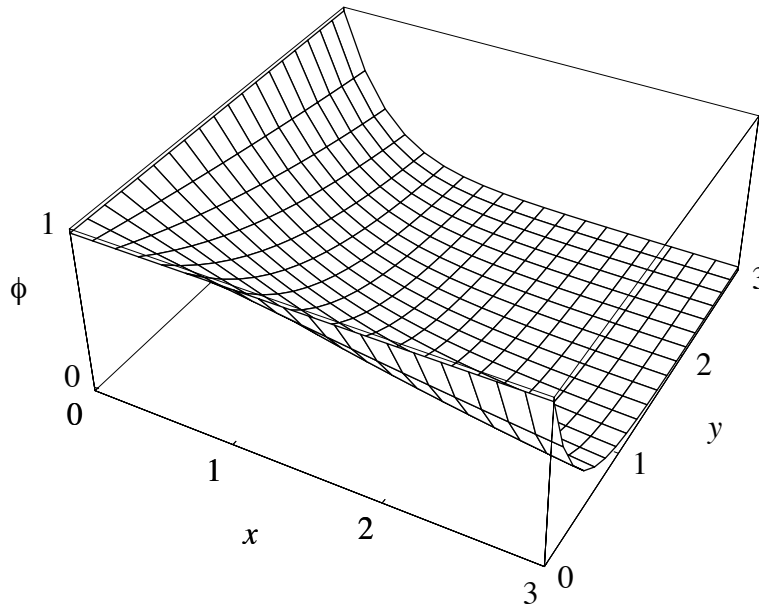
where  $P$  is a polynomial of  $y'$  ( $= dy/dx$ ),  $y$  and  $x$ , *i.e.*  $P$  is holomorphic. This problem was first addressed using differential algebra by Ritt in the 1930s (for an introduction to differential algebra see Kolchin, 1973). Ritt extended algebraic ring theory to differential systems, finding that factorisations of differential polynomials can be identified with invariants of  $P$ . In turn this allows integrating factors to be determined for the problem. Later the theory was extended to include systems of differential polynomial forms (see Mishra, 1993). With the present hydrological context in mind, we must locate potentials that lead to flow path equations that are essentially polynomial (in the sense of equation (4.34)) in form. As will be seen in section 6, simple pumping well systems in two dimensions satisfy this requirement, but more complicated flow potentials do not.

### 5. Proof of Concept - An Exponential Flow Potential

It is instructive to consider the application of the above methods to a simple example potential. Consider a steady, uniform and homogeneous two-dimensional hydrological system that has a potential given by

$$\phi(x, y) = a e^{bxy} \quad (5.1)$$

where  $a$  and  $b$  are constants. Figure 5.1 shows a plot of this potential for  $a = b = 1$ .



**Figure 5.1:** Three-dimensional plot of the exponential potential  $\phi$  defined in equation (5.1).

The resulting flow path differential equation is

$$\frac{dy}{dx} = \frac{x}{y} \quad (5.2)$$

which can be integrated directly to yield

$$y(x) = \pm \sqrt{x^2 + s^2} \quad (5.3)$$

where  $s$  is an integration constant related to the starting point of the flow path. The choice of sign in equation (5.3) is related to the region of space in which the flow path is to be drawn.

As an example of the use of the Frobenius method, described in section 4.1, expanding  $y$  to eighth order in  $x$ , inserting in equation (5.2) and equating coefficients of powers of  $x$  to zero yields

$$a_1 = a_3 = a_5 = \dots = 0$$

$$a_2 = \frac{1}{2a_0}; a_4 = -\frac{1}{8a_0^3}; a_6 = \frac{1}{16a_0^5}; a_8 = -\frac{5}{128a_0^7} \text{ etc} \quad (5.4)$$

The Frobenius form for  $y(x)$  can then be assembled as

$$y(x) = a_0 + \frac{x^2}{2a_0} - \frac{x^4}{8a_0^3} + \frac{x^6}{16a_0^5} - \frac{5x^8}{128a_0^7} + \dots \quad (5.5)$$

The simplification of this solution can then proceed by the hypergeometric function method outlined in section 4.2. The coefficient ratios are

$$\frac{1}{2a_0^2}, -\frac{1}{2a_0^2}, -\frac{3}{2a_0^2}, -\frac{5}{2a_0^2} \quad (5.6)$$

Choosing the polynomial degrees  $p = 2$  and  $q = 1$  yields the identification matrix

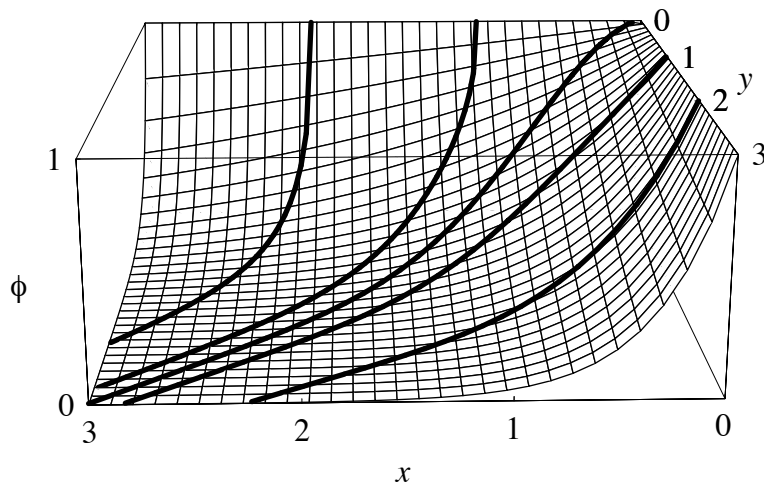
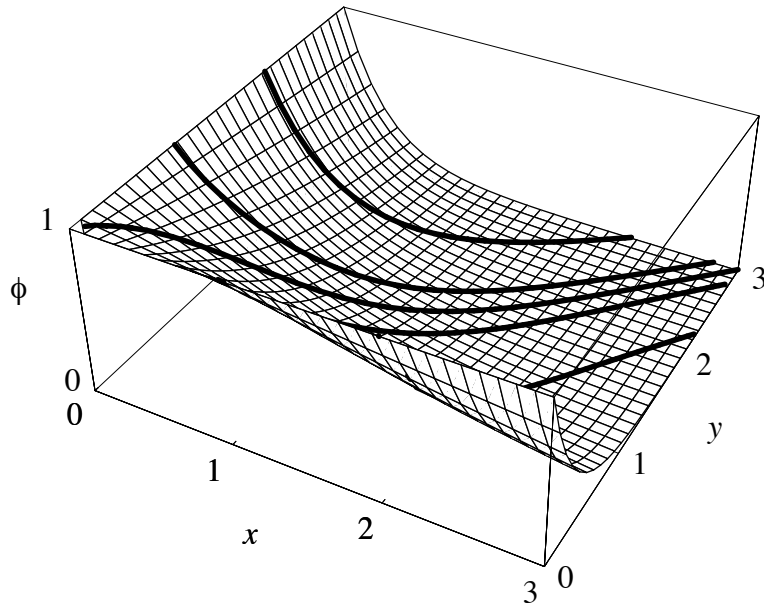
$$\begin{pmatrix} 1 & 1 & 1 & -\frac{1}{2a_0^2} & -\frac{1}{2a_0^2} \\ 1 & 2 & 4 & \frac{1}{a_0^2} & \frac{2}{a_0^2} \\ 1 & 3 & 9 & \frac{9}{2a_0^2} & \frac{27}{2a_0^2} \\ 1 & 4 & 16 & \frac{10}{a_0^2} & \frac{40}{a_0^2} \end{pmatrix} \quad (5.7)$$

The non-zero polynomial coefficients are  $p_0 = 3/(2a_0^2)$  and  $p_1 = -1/a_0^2$ , leading to the summable identification

$$y(x) = a_0 {}_1F_0\left(-\frac{1}{2};; -\left(\frac{x}{a_0}\right)^2\right)$$

$$= \sqrt{x^2 + a_0^2} \quad (5.8)$$

which is in agreement with equation (5.3).



**Figure 5.2:** Two different views of exact flow paths drawn on the exponential potential surface of Figure 5.1.

Expansions of  $y$  to greater than eighth order lead to identical non-zero polynomial coefficients, indicating that the minimal hypergeometric function identification is stable. Figure 5.2 shows flow paths mapped onto the potential (equation (5.1)) from two different view points. Note that the flow paths follow trajectories of steepest descent on the potential surface, *i.e.* they run downhill.

## 6. Pumping Wells in Two Dimensions

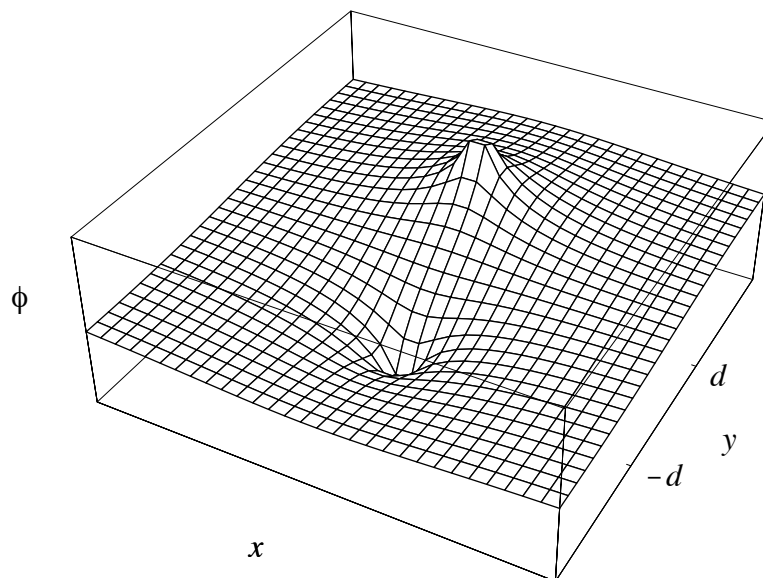
In the initial applications of the technique, pumping well systems in homogeneous two-dimensional aquifers are considered.

### 6.1 Problem 1 - A Well Doublet

Consider a system of two wells, one injecting and one abstracting at matched rates, in a flat two-dimensional aquifer with zero areal recharge. If they are separated by distance  $2d$ , the potential for the system is

$$\phi(x,y) = \text{Log}((y+d)^2 + x^2) - \text{Log}((y-d)^2 + x^2) \quad (6.1)$$

where the  $y$  axis has been chosen to align with the doublet. Figure 6.1 shows a plot of the doublet potential.



**Figure 6.1:** Three-dimensional plot of the doublet well potential. Although the aquifer domain is infinite in the  $x$ - $y$  plane, a boundary frame is drawn to aid the eye.

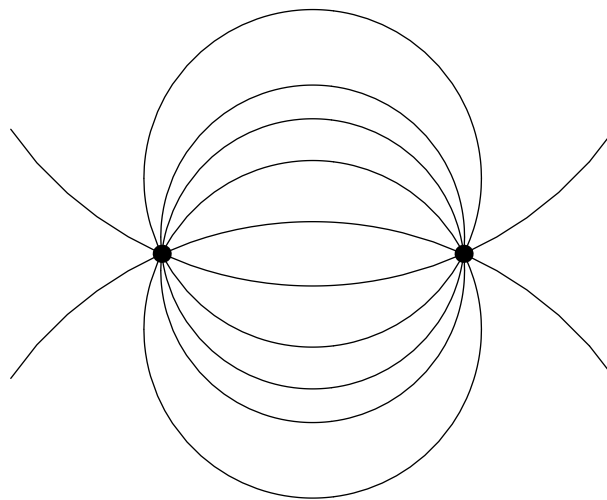
Using equation (2.4), the flow path differential equation for the doublet well system is

$$\frac{dx}{dy} = \frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} = -\frac{2xy}{d^2 + x^2 - y^2} \quad (6.2)$$

Standard integration techniques give

$$x(y) = \pm \frac{1}{2} e^{-s} \left( \pm 1 + \sqrt{1 + 4e^{2s}(d^2 - y^2)} \right) \quad (6.3)$$

where  $s$  is an integration constant and the choices of sign denote region-specific solutions in the  $x$ - $y$  plane. By choosing  $s$  values to match regional solutions at region boundaries, the following flow path plot (Figure 6.2) can be generated.



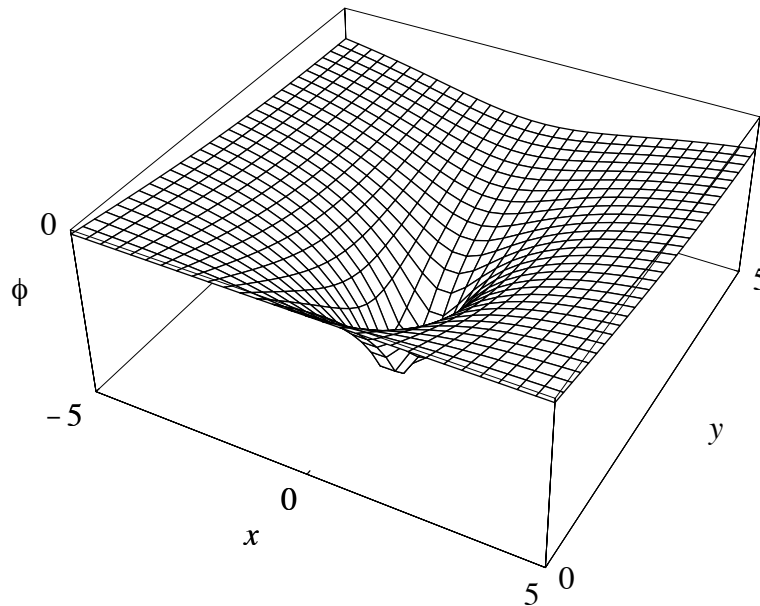
**Figure 6.2:** Flow paths for a balanced well doublet system in 2D plan projection. Since the method of equation (2.4) was used, temporal and velocity information for the flow paths was lost.

## 6.2 Problem 2 - A Well Near a River

Consider a steady system comprised of a river and a nearby pumping well in an aquifer receiving zero areal recharge. If the river is assumed to be a fixed (zero) head condition on the boundary of the aquifer, the corresponding potential is the same as that used for the well doublet case:

$$\phi(x, y) = \text{Log}((y + d)^2 + x^2) - \text{Log}((y - d)^2 + x^2) \quad (6.4)$$

In this case we restrict the solution domain to  $y \geq 0$ , giving the potential shown in Figure 6.3.



**Figure 6.3:** Potential surface for a pumping well centred at  $(x,y) = (0,2)$  near a linear river (at  $y=0$  boundary) with zero potential (fixed head). Apart from the  $y=0$  boundary, the frames drawn in the figure are simply to aid the eye, and do not correspond to physical boundaries.

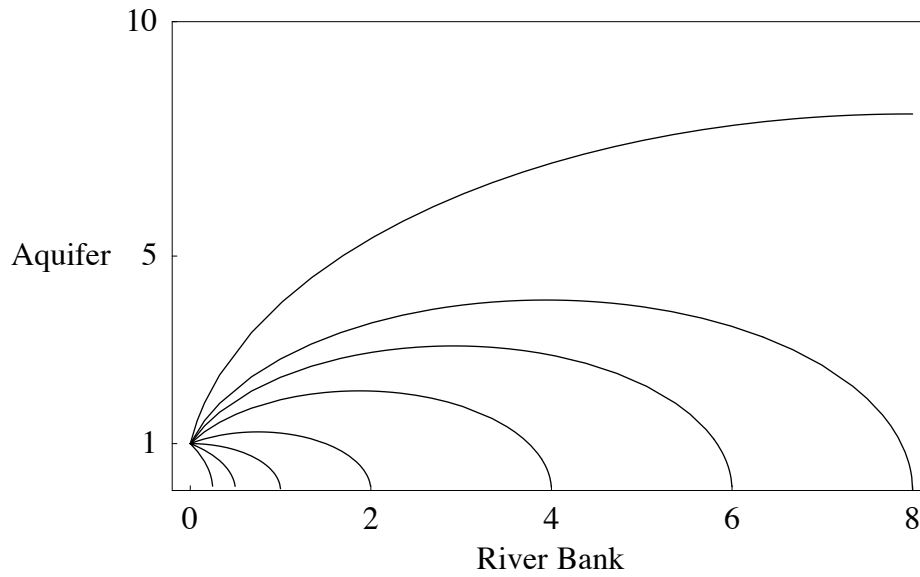
In this case the flow paths are required as functions of  $x$ . The corresponding differential equation is (*c.f.* equation (6.2)):

$$\frac{dy}{dx} = -\frac{d^2 + x^2 - y^2}{2xy} \quad (6.5)$$

which has solutions

$$y(x) = \pm \sqrt{d^2 - x^2 + 2sx} \quad (6.6)$$

where  $s$  is an integration constant which can be related to the river boundary starting position of the flow path. Figure 4 shows a family of flow paths originating at the river boundary and ending at the well. Note that no information as to the seepage rates per unit length from the river bank is included in equation (6.6). This information is lost by choosing the formulation of equation (2.4) over that of equation (2.3).



**Figure 6.4:** Flow paths for a well near a linear river in 2D plan projection. Since the method of equation (2.4) was used, temporal and velocity information for the flow paths was lost. The river bank (aquifer) axis corresponds to the  $x$  ( $y$ ) axis in Figure 6.3.

### 6.3 Problem 3 - A Well in a Regional Flow Field

The flow potential for a well in a regional (uniform) flow field is given by (Bear and Jacobs, 1965)

$$\phi(x, y) = -qy - \frac{Q}{4\pi} \ln(x^2 + y^2) \quad (6.7)$$

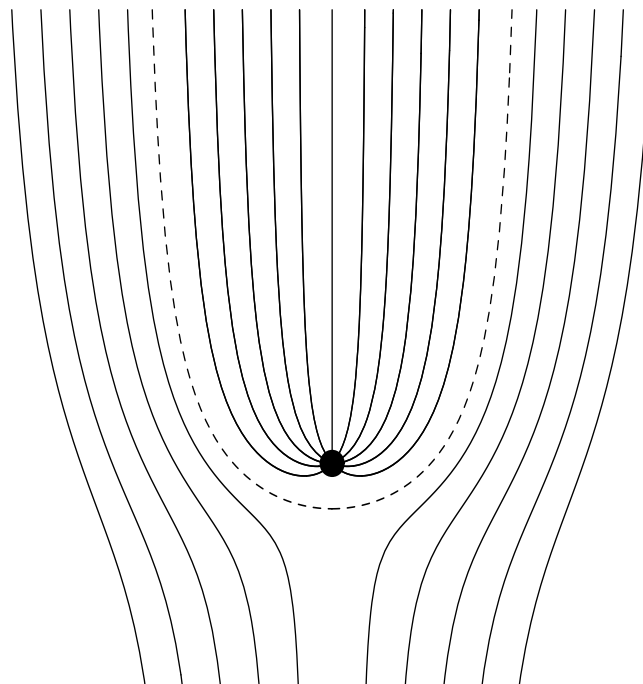
where  $q$  is the regional gradient aligned with the  $y$  axis, and  $Q$  is the discharge of the well per unit thickness of the aquifer. The time-implicit flow path equation is then

$$\frac{dy}{dx} = \frac{y}{x} + \frac{q(x^2 + y^2)}{2Qx} \quad (6.8)$$

which can be integrated by standard techniques to yield

$$y(x) = x \tan\left(\frac{qx - s}{2Q}\right) \quad (6.9)$$

The parameter  $s$  is an integration constant that selects a given flow path. Figure 6.5 shows the flow paths for the case  $q = Q = -1$ , *i.e.* flow from top to bottom. The dividing streamline (dashed) separates flow that is captured by the well from flow that bypasses the well. This capture zone boundary is given by the flow path generated by using  $s = \pi$  in equation (6.9). Exact flow paths for other values of  $q$  and  $Q$  can be generated trivially using modern computer software. Note that the flow paths in Figure 6.5 can also be generated using the complex potential approach of section 4.4. However, the simple system of this section allows us to continue with direct integration methods. Complex potential theory comes into its own where many wells are involved.



**Figure 6.5:** Flow paths for a well in a regional flow field in 2D plan projection. The regional flow is from top to bottom. The dashed line represents the capture zone for the well.

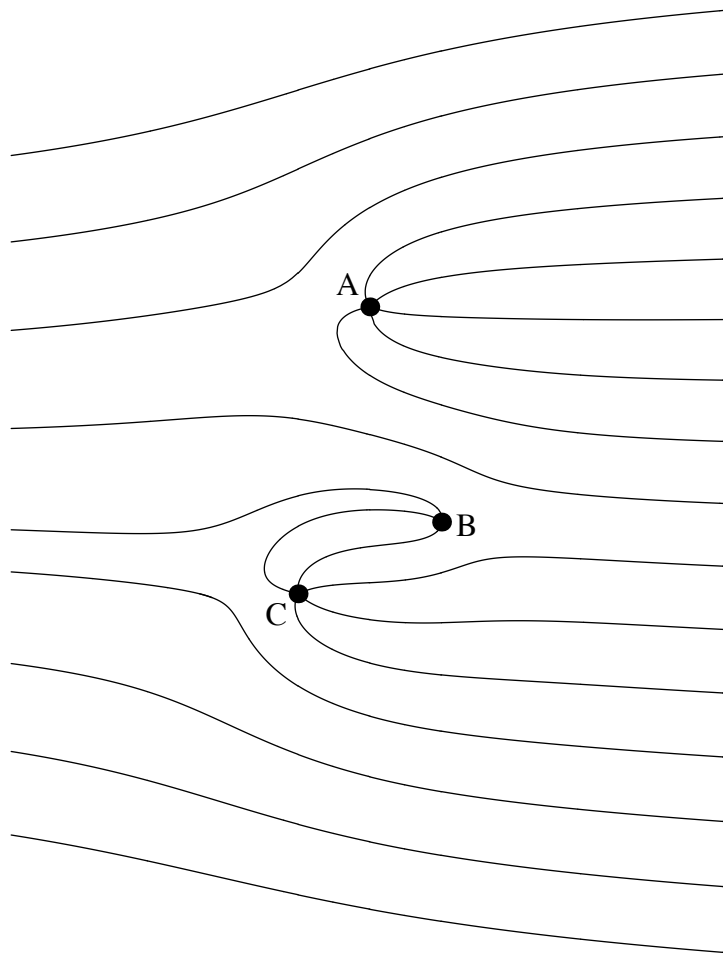
#### 6.4 Problem 3 - $N$ Wells in a Regional Flow Field

The previous problems have concentrated on characteristics of simple dual-well systems. Of more interest to water resource managers is the problem of a finite number,  $N$ , of pumping wells situated in a regional flow field in plan projection. The flow potential for such a problem is

$$\phi(x, y) = qx + \sum_{n=1}^N \frac{Q_n}{4\pi} \ln((x - x_n)^2 + (y - y_n)^2) \quad (6.10)$$

where  $q$  is the regional flow gradient (assumed parallel to the  $x$  axis),  $(x_n, y_n)$  are the locations of the  $N$  wells of strength  $Q_n$ . From section 4.4, the associated complex potential is

$$f(z) = qz + \sum_{n=1}^N \frac{Q_n}{4\pi} \ln(z - z_n) \quad (6.11)$$



**Figure 6.6:** Flow paths for a sample three-well system in a regional flow field in 2D plan projection. The regional flow is from right to left. The extraction wells A and C have equal strengths; injection well B has strength opposite in sign and one-half in magnitude.

where  $z_n = (x_n, y_n)$ , which yields the streamfunction

$$\psi(x, y) = qy + \sum_{n=1}^N \frac{Q_n}{4\pi} \arctan\left(\frac{y - y_n}{x - x_n}\right) \quad (6.12)$$

Flow paths are determined from equation (6.12) by plotting lines of constant  $\psi$ . Figure 6.6 shows arbitrarily chosen flow paths for a regional flow (from right to left) system containing two equal strength extraction wells (A and C) and one half strength injection well (B). Calculating flow nets for practical systems governed by equations (6.10) and/or (6.12) is simply a matter of contouring  $\phi$  (for head) and  $\psi$  (for streamfunction) with the appropriate well strengths and positions. This is straightforward with modern mathematical software.

## 7. Shallow Water Table Lakes

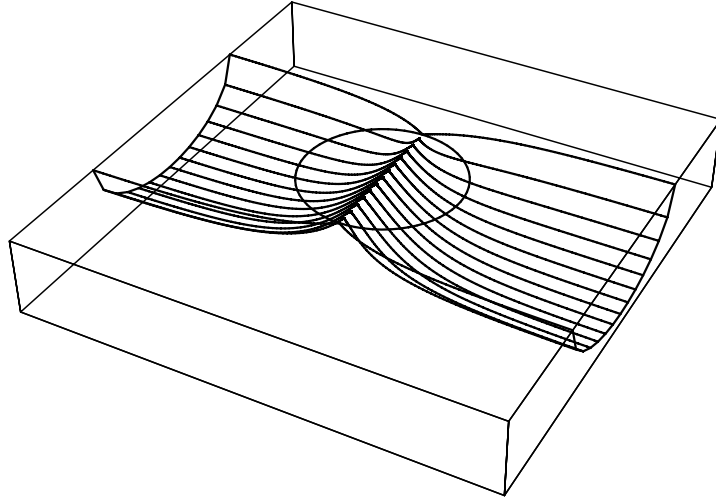
Of particular interest in the Swan Coastal Plain of south-western Western Australia is the interaction of groundwater resources with the numerous shallow lakes and wetlands on the Plain. Many of these lakes and wetlands are seasonal, expressing the fluctuation in water table elevation throughout the year. Recent research on the hydrological cycles of these water bodies (Townley *et al.*, 1993) led to the derivation of an exact groundwater flow potential for a shallow circular lake overlying a saturated unconfined aquifer subject to zero vertical recharge (Trefry and Townley, 1998). The potential, for a lake of radius  $a$ , is

$$\phi_a(x, y, z) = -2Vx \left[ \frac{a\sqrt{\lambda}}{\pi(a^2 + \lambda)} + \frac{1}{\pi} \arctan\left(\frac{\sqrt{\lambda}}{a}\right) \right] \quad (7.1)$$

where

$$\lambda = \frac{1}{2} \left( -a^2 + x^2 + y^2 + z^2 + \sqrt{4a^2z^2 + (-a^2 + x^2 + y^2 + z^2)^2} \right) \quad (7.2)$$

The Cartesian axes are chosen so that the  $x$  coordinate is measured in the direction of regional gradient (slope  $V$ ), and  $z$  is the vertical coordinate. Although the flow paths for this system are strongly three-dimensional, as shown by the approximate numerical flow paths presented in Figure 7.1, critical geometrical flow parameters can be pursued by working in reduced dimensions.



**Figure 7.1:** Three-dimensional flow paths for a sample lake-aquifer system. The bounding box is drawn purely to aid the eye; no boundary conditions are implied other than the zero recharge top face ( $z = 0$ ).

### 7.1 Lake Capture Zone Width

Consider the width of the lake capture/release zone. This is defined by the flow path (lying in the  $z = 0$  plane) that is tangential to the lateral margin of the lake, *i.e.* perpendicular to the regional flow direction. Since the  $z = 0$  plane is a plane of mirror symmetry in the problem, this flow path may be obtained by integrating equations (7.1) and (7.2) for  $z = 0$ , which provides a substantial simplification for  $\lambda$ :

$$\lambda = -a^2 + x^2 + y^2 \quad (7.3)$$

The  $z = 0$  flow paths are then given by the solution of either the time-explicit autonomous system of first-order differential equations (see equation (2.3))

$$\begin{aligned} \frac{dx}{dt} &= -\frac{2a(x^4 + a^2x^2 + 2x^2y^2 + y^4 - a^2y^2)}{\pi(x^2 + y^2)^2 \sqrt{-a^2 + x^2 + y^2}} - \frac{2}{\pi} \operatorname{arc cot} \left( \frac{a}{\sqrt{-a^2 + x^2 + y^2}} \right) \\ \frac{dy}{dt} &= -\frac{4a^3xy}{\pi(x^2 + y^2)^2 \sqrt{-a^2 + x^2 + y^2}} \end{aligned} \quad (7.4)$$

or of the single time-implicit first-order differential equation (see equation (2.4))

$$\frac{dx}{dy} = \frac{x^4 + a^2x^2 + 2x^2y^2 + y^4 - a^2y^2}{2a^2xy} + \frac{(x^2 + y^2)^2 \sqrt{-a^2 + x^2 + y^2} \operatorname{arc cot} \left( \frac{a}{\sqrt{-a^2 + x^2 + y^2}} \right)}{2a^2xy} \quad (7.5)$$

Solution of equations (7.4) or (7.5) will provide exact geometrical information on the lateral shape of lake capture and release zones for zero-recharge flow-through conditions. The solution of these equations has not yet been completed.

## 7.2 Lake Capture Zone Depth

Consider the depth of the lake capture/release zone. This is defined by the flow path (lying in the  $y = 0$  plane) that is tangential to the lake bed at the origin. Since the  $y = 0$  plane is also a plane of mirror symmetry in the problem, this flow path may be obtained by integrating equations (7.1) and (7.2) for  $y = 0$ . In this case  $\lambda$  has the more complicated form:

$$\lambda = \frac{1}{2} \left( -a^2 + x^2 + z^2 + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2} \right) \quad (7.6)$$

The flow path equations in the  $y = 0$  plane are then (time explicit)

$$\begin{aligned} \frac{dx}{dt} = & -\frac{2}{\pi} \operatorname{arc cot} \left( \frac{\sqrt{2}a}{\sqrt{-a^2 + x^2 + z^2} + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}} \right) \\ & + \frac{(a^2 + x^2 + z^2) \sqrt{-a^2 + x^2 + z^2} + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}}{\sqrt{2\pi}ax^2} \\ & - \frac{(a^2 + x^2 + z^2)^2 \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2} \sqrt{-a^2 + x^2 + z^2} + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}}{\sqrt{2\pi}ax^2 (a^2 - 2ax + x^2 + z^2) (a^2 + 2ax + x^2 + z^2)} \end{aligned} \quad (7.7)$$

$$\frac{dz}{dt} = -\frac{8\sqrt{2}a^3xz\left(a^2 + x^2 + z^2 + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}\right)}{\pi\sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}\sqrt{-a^2 + x^2 + z^2}\sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}}$$

and (time implicit)

$$\begin{aligned} \frac{dx}{dz} &= \frac{(a^2 + x^2 + z^2)\sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2} - a^4 + x^4 + z^4 + 2x^2z^2}{4a^2xz} \\ &+ \frac{1}{4\sqrt{2}a^3xz} \operatorname{arc\,cot} \left( \frac{\sqrt{2}a}{\sqrt{-a^2 + x^2 + z^2 + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}}} \right) \\ &\times \sqrt{-a^2 + x^2 + z^2 + \sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2}} \left\{ a^4 - 2a^2x^2 + 2a^2z^2 + x^4 + z^4 + 2x^2z^2 \right. \\ &\left. + (a^2 + x^2 + z^2)\sqrt{4a^2z^2 + (-a^2 + x^2 + z^2)^2} \right\} \end{aligned} \quad (7.8)$$

Solution of equations (7.7) or (7.8) will provide exact geometrical information on the vertical shape of lake capture and release zones for zero-recharge flow-through conditions. The solution of these equations has not yet been completed.

### 7.3 Remarks on Solution Methods

The necessity for advanced analytical techniques for dealing with the flow path equations of sections 7.1 and 7.2 is obvious. The differential systems of equations (7.4) and (7.7) are candidates for solution via the Lie commutation theorem of section 4.3. Equations (7.5) and (7.8) are not autonomous, as they explicitly involve both the independent variable  $y$  (7.5) and  $z$  (7.8) and the dependent variable  $x$ , and series expansion techniques similar to those discussed in sections 4.1 and 4.2 may be best suited to their solution. The Cauchy-Riemann equations, which may be used to infer the complex potential corresponding to equation (7.1), are related to the differentials in equations (7.4) and (7.7). The complexity of these differentials mean that it is unlikely that a suitable complex potential can be found. Finally, none of these equations can be expressed in differential polynomial form, as discussed in section 4.5, ruling out the use of differential algebra techniques.

## 8. Concluding Remarks

This report has described techniques by which exact flow paths can be calculated from algebraic groundwater flow potentials. Since the potentials are essentially unrestricted in form, several approaches to solving the flow path equations have been discussed. For potentials governing the two-dimensional (in plan projection) flow in the vicinity of pumping wells, complex potential theory provides a simple and direct solution of the flow path problem, for quite arbitrary arrangements of the wells. Thus scenarios involving uniform flow and pumping wells are no longer problematic.

Systems involving shallow water table lakes are more troublesome. In these cases the potentials have complicated forms, yielding flow path differential equations that are strongly non-linear. Algebraic solution of the full three-dimensional flow path equations appears unlikely. By taking advantage of natural symmetries of the lake-aquifer problem, reductions in dimensionality are achieved, leading to significant simplifications in the flow path equations. The simplified equations offer the prospect of determining critical geometrical parameters for lake-aquifer flow. For example, by considering flow paths that are tangential to high symmetry points on the lake bed, *i.e.* *principal flow paths*, the limiting capture zone widths and depths noted in previous work may be determined. Lie symmetry group methods appear to provide the best means of attack on these equations, although success is by no means certain. The need for further research on solving the lake-aquifer principal flow paths is indicated.

In pursuing the Lie methods for lake-aquifer systems, the ultimate goal is the determination of theoretically exact capture zone geometries. Such results would provide crucial benchmarks for computational simulations of similar systems, where numerical errors have been shown to be important, and also give simple rules of thumb for water resource managers regarding land use impacts on neighbouring lake and wetland systems.

## 9. Acknowledgment

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